Instability of AdS and simple semilinear wave equations on compact manifolds - numerical studies

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Outline

Part 1 Motivation:

- Conjecture of instability of Anti-de Sitter spacetime
- What is the mechanism of this instability?

Part 2 Trying to gain insight through toy models:

- Cubic wave equation on a torus
- Yang-Mills equation on the Einstein universe
- Wave map from the Einstein universe to the 3-sphere
- Wave-map type equations on AdS

Conclusions

Anti-de Sitter (AdS) spacetime in $d+1$ dimensions

- AdS is the maximally symmetric solution of the vacuum Einstein equations $R_{\alpha\beta} = \lambda g_{\alpha\beta}$ with negative λ

$$g = -(1 + r^2/\ell^2)dt^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\Omega^2_{S^{d-1}}$$

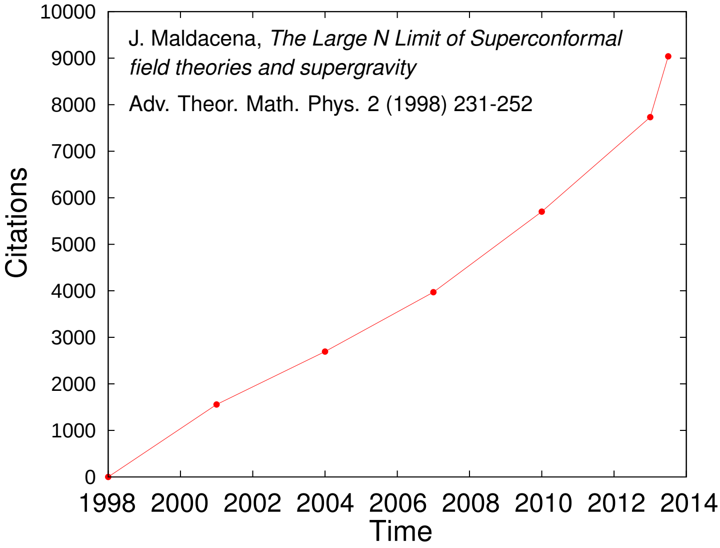
  where $\ell^2 = -d/\lambda$, $r \geq 0$, and $-\infty < t < \infty$.

- Substituting $r = \ell \tan x$ $(0 \leq x < \pi/2)$ we get

$$g = \frac{\ell^2}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x\, d\Omega^2_{S^{d-1}}\right)$$

- Conformal infinity $x = \pi/2$ is the timelike cylinder $\mathscr{I} = \mathbb{R} \times S^{d-1}$
- Null geodesics reach $\mathscr{I}$ in finite time so AdS is effectively bounded

AdS is the most popular spacetime on the arXiv (due to AdS/CFT)



Is AdS stable?

- By the positive energy theorem AdS space is the unique ground state among asymptotically AdS spacetimes (much as Minkowski space is the unique ground state among asymptotically flat spacetimes)
- Minkowski spacetime was proved to be asymptotically stable by Christodoulou and Klainerman (1993)
- Key difference between Minkowski and AdS: **the main mechanism of stability of Minkowski - dissipation of energy by dispersion - is absent in AdS** (for no flux boundary conditions $\mathscr{I}$ acts as a mirror)
- The problem of stability of AdS has not been explored until recently; notable exception: proof of local well-posedness by Friedrich (1995)
- The problem seems tractable only in spherical symmetry so one needs to add matter to generate dynamics. Simple choice: a massless scalar field

AdS gravity with a spherically symmetric scalar field

Conjecture (B-Rostworowski 2011)

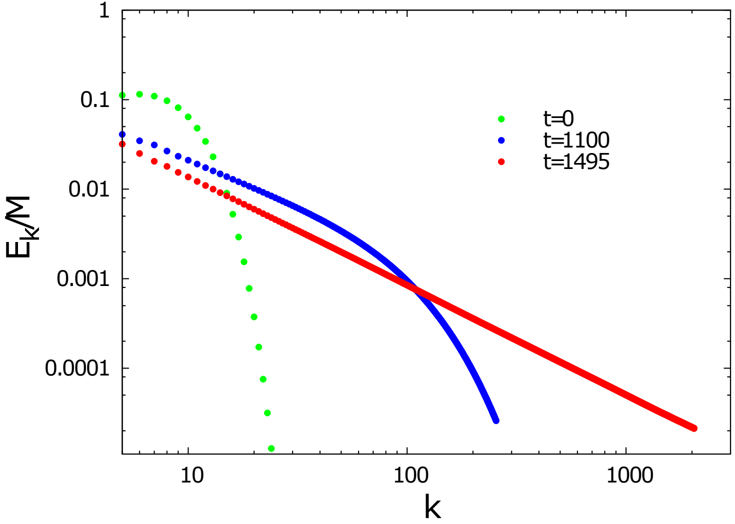
AdS$_{d+1}$ (for $d \geq 3$) is unstable under arbitrarily small perturbations

Heuristic picture (supported by a nonlinear perturbation analysis and numerical evidence): **due to resonant interactions between harmonics the energy is transferred from low to high frequencies.** The concentration of energy on finer and finer scales inevitably leads to the formation of a horizon (the endstate of instability is the Schwarzschild-AdS black hole)

Some follow-up studies:

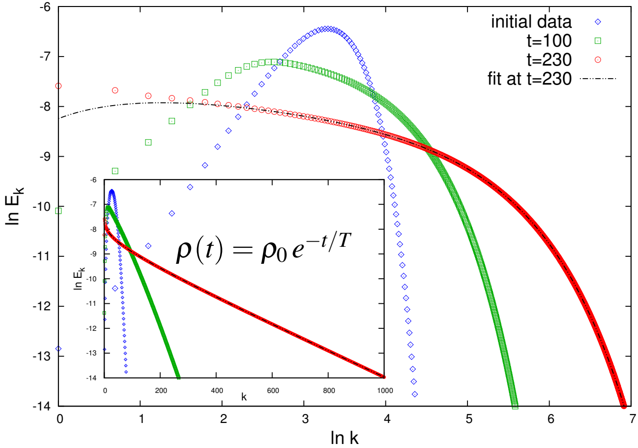
- The turbulent instability is absent for some perturbations, in particular there is analytic and numerical evidence for the existence of **stable time-periodic solutions** (Maliborski-Rostworowski 2013)
- In $2+1$ dimensions small perturbations of AdS$_3$ remain smooth for all times but their **radius of analyticity shrinks to zero as** $t \to \infty$ and higher Sobolev norms grow unbounded (B-Jałmużna 2013)

Energy spectrum for $d = 3$



Just before collapse $E_k \sim k^{-\alpha}$ with $\alpha \approx 1.2$ $(6/5??)$

Energy spectrum for $d = 2$



$$E_k(t) = C(t)\, k^{-\alpha(t)} e^{-2\rho(t)k}$$

Part 2

Trying to gain insight into the dynamics of asymptotically AdS spacetimes through simple nonlinear wave equations on spatially bounded domains

Motto: *The more he looked inside the more Piglet wasn't there.*

Winnie-the -Pooh

Nonlinear waves on bounded domains

- Example: $u_{tt} - \Delta u + u^3 = 0$ for $u(t,x)$ with $x \in M$ (compact manifold)
- Due to the lack of dispersion the long-time dynamics is much more complex and mathematically challenging than in the non-compact setting
- The main goal: understand out-of-equilibrium dynamics of small solutions
- Is the ground state $u = 0$ stable (say in H_2 norm)?
- This is an open problem even for $u_{tt} - u_{xx} + u^3 = 0$ on S^1 **!**
- Key enemy: **weak turbulence** - transfer of energy to progressively smaller scales (gradual loss of smoothness as $t \to \infty$)

General strategy for small initial data

- Let $e_n(x)$ and ω_n^2 be the eigenfunctions and eigenvalues of $-\Delta$ on M
- Decompose $u(t,x) = \varepsilon \sum_n a_n(t) e_n(x)$ and rewrite the equation on the Fourier side as an infinite dimensional dynamical system

$$\ddot{a}_n + \omega_n^2 a_n = \varepsilon^2 \sum c_{jkm}^n a_j a_k a_m, \qquad c_{jkm}^n = (e_j e_k e_m, e_n)$$

- The entire information about the dynamics is contained in the frequencies ω_n and the interaction coefficients c_{jkm}^n
- Are there non-trivial resonances ($\omega_n = \pm\omega_j \pm \omega_k \pm \omega_m$ for $c_{jkm}^n \neq 0$)?
  - If not: try to construct the solutions perturbatively (for example, using the method of normal forms). Main difficulty: small divisors.
  - If yes: drop all non-resonant terms and hope that the remaining resonant system is amenable to mathematical analysis
- Key object of interest: evolution of the energy spectrum $E_n(t) = \dot{a}_n^2 + \omega_n^2 a_n^2$. The transfer of energy to high fequencies can be measured by Sobolev norms $\|u(t)\|_s = \left(\sum \omega_n^{2s} a_n^2\right)^{1/2}$ with $s > 1$.

Example: cubic Klein-Gordon equation on S^1

- Plugging $u(t,x) = \varepsilon \sum_{n\in\mathbb{Z}} a_n(t) e^{inx}$ into $u_{tt} - u_{xx} + \mu^2 u + |u|^2 u = 0$ gives

$$\ddot{a}_n + \omega_n^2 a_n = -\varepsilon^2 \sum_{j-k+m=n} a_j \bar{a}_k a_m$$

- Interaction picture (variation of constants)

$$a_n = a_n^+(t) e^{i\omega_n t} + a_n^-(t) e^{-i\omega_n t}, \quad \dot{a}_n = i\omega_n \left(a_n^+(t) e^{i\omega_n t} - a_n^-(t) e^{-i\omega_n t} \right)$$

leads to the first order system ($\Omega = \pm\omega_j \pm \omega_k \pm \omega_m \mp \omega_n$)

$$\pm 2in\, \dot{a}_n^\pm = \varepsilon^2 \sum_{\substack{j-k+m=n \\ \text{permutations of } \pm}} a_j^\pm \bar{a}_k^\pm a_m^\pm e^{i\Omega t}$$

- Resonant terms correspond to $\Omega = 0$ and $j - k + m = n$. For nonzero mass μ there are no exact resonances ($\omega_n = \sqrt{n^2 + \mu^2}$). For $\mu = 0$, after dropping all non-resonant terms, one gets the resonant system

$$\boxed{\pm 2in\, \dot{a}_n^\pm = \varepsilon^2 \sum_{j-k+m=n} a_j^\pm \bar{a}_k^\pm a_m^\pm + 2\varepsilon^2 \left(\sum_k |a_k^\mp|^2 \right) a_n^\pm}$$

Numerical results

- We solve numerically $u_{tt} - u_{xx} + \mu^2 u + u^3 = 0$ on the interval $-1 \leq x \leq 1$ with periodic boundary conditions for various initial data ▸ Start movie
- For (small) initial data, after a very short time we observe the formation of a coherent structure with the exponentially decaying energy spectrum $E_k(t) \sim e^{-2\rho(t)k}$. The radius of analyticity $\rho(t)$ quickly stabilizes at some (approximately) constant value (the Sobolev norms stop growing)
- Suprisingly, the dynamics for $\mu = 0$ and $\mu \neq 0$ are similar ▸ Start movie
- Analogous behaviour in higher dimensions ▸ Start movie
- It is conceivable that this coherent structure is a transient metastable state with an extremely long lifetime (cf. the Fermi-Pasta-Ulam paradox)
- What is the mechanism of the saturation of the energy transfer?

Yang-Mills on the Einstein universe

- Manifold $M = \mathbb{R} \times S^3$ with the metric

$$g = -dt^2 + dx^2 + \sin^2 x \left(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2 \right)$$

- Spherically symmetric (magnetic) ansatz for the SU(2) Yang-Mills connection

$$A = W(t,x) \tau_1 d\vartheta + \left(\cot\vartheta \, \tau_3 + W(t,x) \tau_2 \right) \sin\vartheta \, d\varphi$$

- The YM equations $\nabla_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0$ reduce to

$$\boxed{W_{tt} = W_{xx} + \frac{W(1-W^2)}{\sin^2 x}}$$

- For smooth initial data the solutions remain smooth for all times (Choquet-Bruhat 1989, Chruściel-Shatah 1997)

- The conserved energy $E = \int_0^\pi \left(W_t^2 + W_x^2 + \frac{(1-W^2)^2}{2\sin^2 x} \right) dx$
- $W(t,0) = \pm 1$ and $W(t,\pi) = \pm 1 \Rightarrow$ two topological sectors $N = 0, 1$.
- In each sector there is a unique static solution:
  $W_0 = 1$ (vacuum) with $E = 0$ and $W_1 = \cos x$ (kink) with $E = 3\pi/4$.
- Linearized perturbations $u = W - W_N$ around the static solutions satisfy

$$u_{tt} + Lu = 0\,, \qquad L = -\frac{d^2}{dx^2} + \frac{3W_N^2 - 1}{\sin^2 x}$$

  The operator L is essentially self-adjoint on $L^2([0,\pi], dx)$.
- The eigenvalues and (orthonormal) eigenfunctions of L are $(k = 0, 1, ...)$

$$\omega_k^2 = (2+k)^2 \quad \text{for} \quad N = 0 \quad \text{and} \quad \omega_k^2 = (2+k)^2 - 3 \quad \text{for} \quad N = 1$$

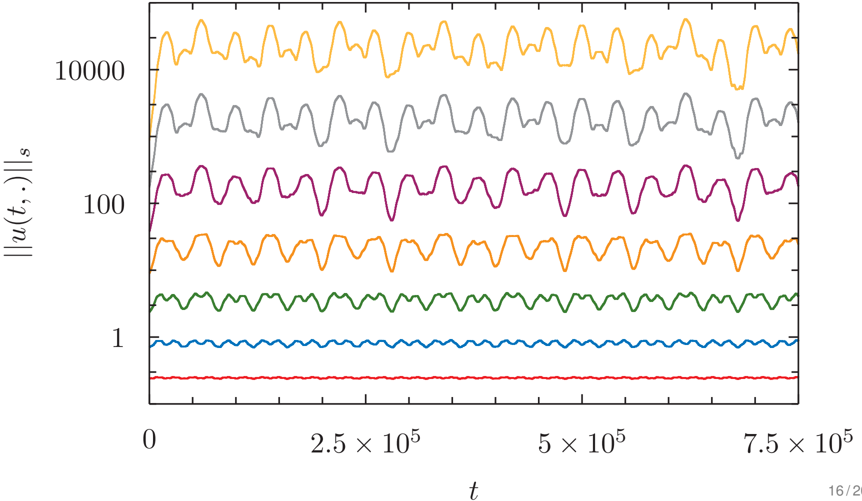
$$e_0 = \sqrt{\tfrac{8}{3\pi}} \sin^2 x,\ e_1 = \sqrt{\tfrac{16}{\pi}} \sin^2 x \cos x,\ e_2 = \sqrt{\tfrac{32}{15\pi}} \sin^2 x\,(6\cos^2 x - 1), \ldots.$$

Numerical results

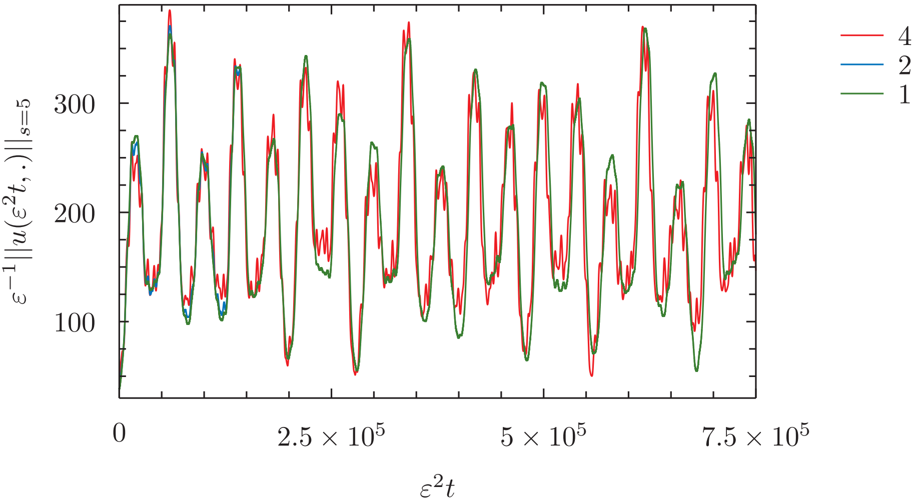
- Transfer of energy to high frequencies gets frozen after some time

  Start movie

- Sobolev norms ($s = 1, ..., 7$) for a gaussian perturbation of W_0



Evidence for (meta)stability of W_0



The scaling of $||u(t)||_5$ with respect to the amplitude ε of the gaussian.

Equivariant wave maps from $\mathbb{R} \times S^3$ into S^3

$$U_{tt} = U_{xx} + \frac{2\cos x}{\sin x} U_x - \frac{\sin(2U)}{\sin^2 x}$$

- Self-similar blowup $U_x(t,0) \sim b(T-t)^{-1}$ as $t \nearrow T$ for large data; the same as in Minkowski space (blowup does not see the curvature)
- Is there a threshold for blowup? One may speculate that the lack of dispersion combined with the supercritical scaling of energy can lead to blowup for arbitrarily small perturbations (as in the case of AdS)
- Numerical simulations do not support this speculation: it seems that for generic one-parameter families of initial data $U(0,x) = \varepsilon f(x)$ there is a critical amplitude ε^* below which the solutions are globally regular in time
- The linear spectrum is not resonant. Is this fact responsible for the existence of threshold for blowup?

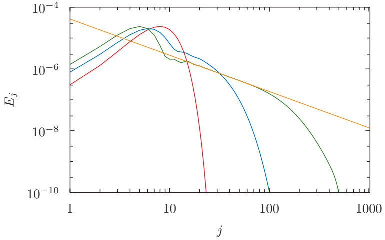
Wave-map type equations on AdS

$$U_{tt} = U_{xx} + \frac{d-1}{\cos x \sin x} U_x - \frac{F(U)}{\sin^2 x}$$

$$(t,x) \in \mathbb{R} \times [0, \pi/2)$$

Example: $d = 4$, $F(U) = \frac{4}{3}(e^{-2U} - e^{-8U})$

- Fully resonant spectrum: $\omega_k = 6 + 2k$
- Formation of the energy cascade with a power-law spectrum and the blowup of $U_{xx}(t,0)$ in a finite time T.



- For initial data of size ε, it appears that $T(\varepsilon) \sim \varepsilon^{-2}$
- Warning: more precise numerical simulations are needed to confirm these preliminary observations and formulate convincing conjectures

Conclusions

- Dynamics of asymptotically AdS spacetimes is an interesting meeting point of basic problems in general relativity and PDE theory.
- Understanding of an out-of-equilibrium dynamics of small solutions is mathematically challenging even for the simplest nonlinear wave equations on compact manifolds, let alone the Einstein equations.
- Most of the above simple models exhibit a qualitatively different behaviour than Einstein-AdS equations so they are not good toy models. Nonetheless, they help us understand how special Einstein's equations are (and they are interesting on their own).
- Wave maps on AdS seem to be a good playground for gaining insight into the turbulent instability of AdS