From AdS to BEC

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Spatially confined Hamiltonian systems

- Evolution of nonlinear waves on unbounded domains is stabilized by the dissipation of energy by radiation
- For spatially confined systems this stabilizing mechanism is absent and the nonlinear self-interactions of waves remain important for all times, inducing complicated energy transfer patterns.
- How the energy injected into the system gets distributed over the degrees of freedom during the evolution?
- Can the energy flow to arbitrarily high frequencies (*weak turbulence*)?
- Despite recent progress, the question of weak turbulence remains unanswered even for very simple confined Hamiltonian systems
- I will discuss systems with three different types of confinement: compactness of the domain, trapping potential, and the time-like boundary at infinity.

General strategy

- For a spatially confined system, the associated linearized system has a purely discrete spectrum of frequencies
- Expanding solutions in the basis of linear eigenstates one transforms the original PDE into an infinite-dimensional dynamical system with discrete degrees of freedom ('modes').
- The nonlinearity generates new frequencies that may lead to resonances between the modes. The resonances dominate the transfer of energy.
- Dropping all nonresonant terms from the Hamiltonian one obtains a simplified infinite-dimensional dynamical system, called the resonant system, which accurately approximates the dynamics of small amplitude solutions of the original PDE on long time scales
- Strategy: try to understand the dynamics of the resonant system and then export this knowledge to the original PDE.

Example

- Background geometry: the Einstein cylinder $\mathscr{M} = \mathbb{R} \times \mathbb{S}^3$ with metric

$$g = -dt^2 + dx^2 + \sin^2 x\, d\omega^2, \qquad (t, x, \omega) \in \mathbb{R} \times [0, \pi] \times \mathbb{S}^2$$

  This spacetime has constant scalar curvature $R(g) = 6$.

- On $\mathscr{M}$ we consider a real scalar field ϕ satisfying

$$\left(\Box_g - \frac{1}{6} R(g)\right) \phi - \phi^3 = \Box_g \phi - \phi - \phi^3 = 0\,.$$

- We assume that $\phi = \phi(t, x)$. Then, $v(t, x) = \sin(x) \phi(t, x)$ satisfies

$$v_{tt} - v_{xx} + \frac{v^3}{\sin^2 x} = 0$$

  with Dirichlet boundary conditions $v(t, 0) = v(t, \pi) = 0$.

- Linear eigenstates: $e_n(x) = \sqrt{\frac{2}{\pi}} \sin(\omega_n x)$ with $\omega_n = n + 1$ $(n = 0, 1, 2, \ldots)$

Time averaging

- Expanding $v(t,x) = \sum_{n=0}^{\infty} c_n(t) e_n(x)$ we get

$$\frac{d^2 c_n}{dt^2} + \omega_n^2 c_n = -\sum_{jkl} S_{njkl}\, c_j c_k c_l, \quad S_{jkln} = \int_0^{\pi} \frac{dx}{\sin^2 x} e_n(x) e_j(x) e_k(x) e_l(x)$$

- Using variation of constants

$$c_n = \beta_n e^{i\omega_n t} + \bar{\beta}_n e^{-i\omega_n t}, \qquad \frac{dc_n}{dt} = i\omega_n \left(\beta_n e^{i\omega_n t} - \bar{\beta}_n e^{-i\omega_n t}\right)$$

  we factor out fast oscillations

$$2i\omega_n \frac{d\beta_n}{dt} = -\sum_{jkl} S_{njkl}\, c_j c_k c_l\, e^{-i\omega_n t}$$

- Each term in the sum has a factor $e^{-i\Omega t}$, where $\Omega = \omega_n \pm \omega_j \pm \omega_k \pm \omega_l$. The terms with $\Omega = 0$ correspond to resonant interactions.
- Let $\tau = \varepsilon^2 t$ and $\beta_n(t) = \varepsilon \alpha_n(\tau)$. For $\varepsilon \to 0$ the non-resonant terms $\propto e^{-i\Omega\tau/\varepsilon^2}$ are highly oscillatory and therefore negligible.

Resonant system

- Keeping only the resonant terms (and rescaling), we obtain (B-Craps-Evnin-Hunik-Luyten-Maliborski, 2016)

$$i\omega_n \frac{d\alpha_n}{d\tau} = \sum_{j=0}^{\infty}\sum_{k=0}^{n+j} S_{njk,n+j-k}\,\bar{\alpha}_j\alpha_k\alpha_{n+j-k}\,,$$

  where $S_{njk,n+j-k} = \min\{n,j,k,n+j-k\}+1$.

- This system (called the conformal flow) provides an accurate approximation to the cubic wave equation on the timescale $\sim \varepsilon^{-2}$.

- This is a Hamiltonian system

$$i\omega_n \frac{d\alpha_n}{d\tau} = \frac{1}{2}\frac{\partial H}{\partial \bar{\alpha}_n}$$

  with

$$H = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n+j} S_{njk,n+j-k}\bar{\alpha}_n\bar{\alpha}_j\alpha_k\alpha_{n+j-k}$$

Other Hamiltonian systems of the form

$$i\omega_n \frac{d\alpha_n}{d\tau} = \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{njk,n+j-k}\, \bar{\alpha}_j \alpha_k \alpha_{n+j-k}$$

- Cubic Szegő equation

$$\omega_n = 1, \qquad S_{njk,n+j-k} = 1$$

  designed and studied by Gérard-Grellier (2010-2016)

- LLL equation (resonant system for the Gross-Pitaevskii equation)

$$\omega_n = 1, \qquad S_{njk,n+j-k} = \frac{(n+j)!}{2^{n+j}\sqrt{n!j!k!(n+j-k)!}}$$

  Germain-Hani-Thomann (2015)

- Resonant system for scalar perturbations of AdS_{d+1} spacetime

$$\omega_n = 2n+d, \qquad S_{njk,n+j-k} \text{ are very complicated}$$

  Balasubramanian et al., Craps-Evnin-Vanhoof (2014)

Basic properties of the systems of the form

$$i\omega_n \frac{d\alpha_n}{d\tau} = \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{njk,n+j-k}\, \bar{\alpha}_j \alpha_k \alpha_{n+j-k}$$

- Symmetries

$$\begin{aligned} \text{Scaling:} \qquad & \alpha_n(t) \to \varepsilon \alpha_n(\varepsilon^2 t) \\ \text{Global phase shift:} \qquad & \alpha_n(t) \to e^{i\theta} \alpha_n(t) \\ \text{Local phase shift:} \qquad & \alpha_n(t) \to e^{in\theta} \alpha_n(t) \end{aligned}$$

- Conserved quantities

$$Q = \sum_{n=0}^{\infty} \omega_n |\alpha_n|^2, \qquad J = \sum_{n=0}^{\infty} n\omega_n |\alpha_n|^2$$

- The Szegő, conformal, and LLL flows are locally (and therefore also globally) well-posed for initial data with finite J.
- For Einstein-scalar-AdS resonant system there is evidence that solutions may become singular in finite time.

Finite-dimensional invariant manifolds

- For one-mode initial data $\alpha_n(0) = \delta_{nN}$, the solution is $\alpha_n(\tau) = \delta_{nN} e^{-i\lambda_N \tau}$
- Three-dimensional invariant manifolds: $\alpha_0 = b$ and for $n \geq 1$

$$\alpha_n = \begin{cases} (bp+a)p^{n-1} & \text{Szegő flow} \\ (bp+an)p^{n-1} & \text{conformal flow} \\ \frac{1}{\sqrt{n!}}(bp+an)\,p^{n-1} & \text{LLL flow} \end{cases}$$

  where a, b, p are functions of τ.

- The dynamics of these invariant manifolds is described by the reduced Hamiltonian system

$$\frac{da}{d\tau} = f_1(a,b,p), \quad \frac{db}{d\tau} = f_2(a,b,p), \quad \frac{dp}{d\tau} = f_3(a,b,p)$$

- Since there are three conserved quanities Q, J, and H (that are in involution), the reduced systems are completely integrable.

Example 1: reduced system for the conformal flow

- The reduced system reads (using $y = \dfrac{|p|^2}{1-|p|^2}$)

$$\frac{\dot p}{(1+y)^2} = \frac{p}{6}\left(2y|a|^2 + \bar b a\right)$$
$$\frac{\dot a}{(1+y)^2} = \frac{a}{6}\left(5|b|^2 + (18y^2+4y)|a|^2 + (6y-1)\bar b a + 10\bar a b\right)$$
$$\frac{\dot b}{(1+y)^2} = b\left(|b|^2 + (6y^2+2y)|a|^2 + b\bar a\right) + a\left(|b|^2 + (4y+2)y^2|a|^2 + y^2\bar b a\right)$$

- Solution

$$y(t) = B + A\sin(\Omega t + \psi), \qquad \Omega = \frac{1}{6}\left(7Q^2 - 6H\right)^{1/2}$$

- The turning points $y_\pm = B \pm A$ provide lower and upper bounds for the inverse and direct cascades of energy, respectively.

- It is easy to show that y_+ is uniformly bounded from above.

Example 2: two-mode initial data for the cubic Szegő flow

- The two-mode initial data $\vec{\alpha}(0) = (\varepsilon, 1, 0, \ldots)$ correspond to

$$b(0) = \varepsilon, \quad a(0) = 1, \quad p(0) = 0$$

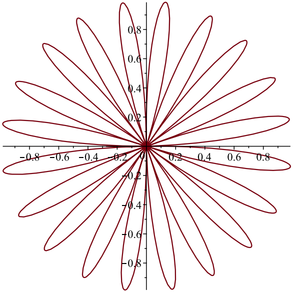
- For the conformal and LLL flows the solution stays close to the stationary state $\vec{\alpha}(\tau) = (0, 1, 0, \ldots)\, e^{-i\lambda\tau}$

- For cubic Szegő equation

$$p(\tau) = -\frac{i}{\sqrt{1+\varepsilon^2/4}}\, \sin(\omega\tau)\, e^{-\frac{1}{2}i\varepsilon^2\tau}$$

with $\omega = \varepsilon\sqrt{1+\varepsilon^2/4}$.

- Thus, $|p(\tau_n)| \sim 1 - \varepsilon^2/8$ for a sequence of times $\tau_n = \frac{(2n+1)\pi}{2\omega}$.



Gérard-Grellier daisy

- This instability provided a hint for the existence of unbounded orbits (Gérard-Grellier, 2015)

Lowest Landau Level equation

- 2d Gross-Pitaevskii equation with isotropic harmonic potential

$$i\partial_t \Psi = \frac{1}{2}\left(-\partial_x^2 - \partial_y^2 + x^2 + y^2\right)\Psi + g|\Psi|^2\Psi$$

  is a mean field model for the Bose-Einstein condensate.

- General solution of the linear problem ($g = 0$)

$$\Psi(t, r, \phi) = \sum_{nm} \alpha_{nm}\, e^{-iE_n t} e^{im\phi} \chi_{nm}(r)$$

  where $e^{im\phi}\chi_{nm}(r)$ are normalized eigenstates of energy $E_n = n + 1$ and angular momentum $m \in \{-n, -n+2, ..., n-2, n\}$.

- The lowest Landau level (LLL) consists of modes with $m = n$:

$$\chi_n(z) = \frac{z^n}{\sqrt{n!}} e^{-\frac{1}{2}|z|^2}, \qquad z = x + iy$$

- The general LLL wavefunction in the frame rotating with angular velocity 1 (where centrifugal and harmonic forces are balanced) is (here $\tau = gt$)

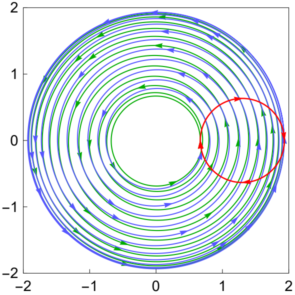
$$\psi(\tau, z) := e^{it}\Psi(t, e^{it}z) = \sum_{n=0}^{\infty} \alpha_n(\tau)\chi_n(z),$$

Vortices in BEC

- A remarkable feature of BEC is the nucleation of quantized vortices when the condensate is stirred above a certain critical angular velocity
- The 3-dimensional invariant manifold of the LLL flow corresponds to single-vortex configurations

$$\psi(\tau,z) = (b(\tau) + a(\tau)z)\, e^{p(\tau)z}\, e^{-\frac{1}{2}|z|^2}$$

- The generic explicit solution represents periodically modulated precession of the vortex
- Such solutions have been seen in experiments!
- It would be very interesting to extend this approach to multi-vortex configurations



Biasi-B-Craps-Evnin, 2017

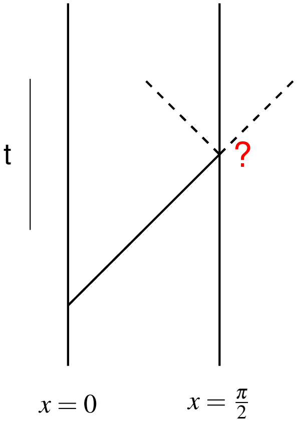
Anti-de Sitter spacetime in $d+1$ dimensions

Manifold $\mathscr{M} = \{t \in \mathbb{R}, x \in [0, \pi/2), \omega \in S^{d-1}\}$ with metric

$$g = \frac{l^2}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x\, d\omega^2_{S^{d-1}}\right)$$

Solution of vacuum Einstein's equations $R_{\alpha\beta} = \lambda g_{\alpha\beta}$ with $\lambda = -d/l^2$.

- Spatial infinity $x = \pi/2$ is the timelike cylinder $\mathscr{I} = \mathbb{R} \times S^{d-1}$ with the boundary metric $ds^2_{\mathscr{I}} = -dt^2 + d\Omega^2_{S^{d-1}}$
- Null geodesics get to infinity in finite time
- Asymptotically AdS spacetimes by definition have the same conformal boundary as AdS
- AdS space is the unique ground state among asymptotically AdS spacetimes.



Conjecture (B-Rostworowski 2011)

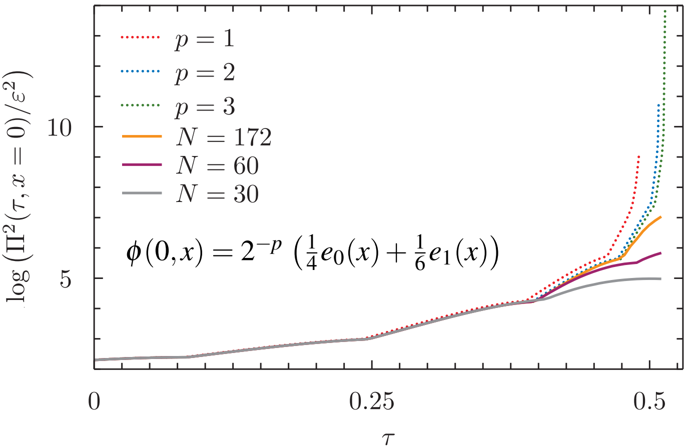
AdS_{d+1}, as the solution of the Einstein-massless-scalar field equations with negative cosmological constant in $d+1$ dimensions (for $d \geq 3$), is unstable under arbitrarily small generic perturbations.

Arguments:

- The linear spectrum is fully resonant. Nonlinear interactions between harmonics give rise to transfer of energy from low to high frequencies.
- The turbulent cascade leads to concentration of energy on finer and finer spatial scales so eventually a black hole is expected to form.
- Numerical evidence: perturbations of size ε collapse in time $\mathcal{O}(\varepsilon^{-2})$.

Resonant approximation (B-Maliborski-Rostworowski, 2015):

- For large n we have $|\alpha_n(\tau)| \sim e^{-\rho(\tau)n}$, where ρ is the analyticity radius. If $\lim_{\tau\to\tau_*} \rho(\tau) = 0$ for some $\tau_* < \infty$ then the solution becomes singular (analyticity strip method Sulem-Sulem-Frisch '83).
- Using mixed analytic-numerical methods we showed that for two-mode initial data $\rho(\tau)$ hits zero in finite time τ_* and $\dfrac{d}{d\tau}\arg(\alpha_n) \sim \ln(\tau_* - \tau)$.



- The instability is captured by the resonant approximation!
- Recently, Moschidis proved instability of AdS for the Einstein-null dust with the inner mirror. In his proof resonances appear to play no role.

From Klein-Gordon on AdS to Gross-Pitaevskii

- AdS metric

$$g = -(1+\frac{r^2}{l^2})dt^2 + \frac{dr^2}{1+\frac{r^2}{l^2}} + r^2 d\omega^2_{\mathbb{S}^{d-1}}$$

- Cubic Klein-Gordon equation on AdS

$$(\Box_g - 1)\,\phi - |\phi|^2\phi = 0$$

- Substituting

$$\phi(t,r,\omega) = \frac{1}{\sqrt{l}}\, e^{-it}\,\Psi\left(\frac{t}{l}, \frac{r}{\sqrt{l}}, \omega\right)$$

and taking the limit $l \to \infty$ we get the Gross-Pitaevskii equation

$$i\partial_t \Psi = -\frac{1}{2}\Delta_{\mathbb{R}^d}\Psi + \frac{1}{2}r^2\Psi + |\Psi|^2\Psi,$$

(O. Evnin, private communication)