Globally regular instability of AdS_3

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Outline

- Instability of AdS_{d+1} for $d \geq 3$
- Why $d = 2$ is different
- Analyticity strip method
- Evidence for weak turbulence
- Open questions

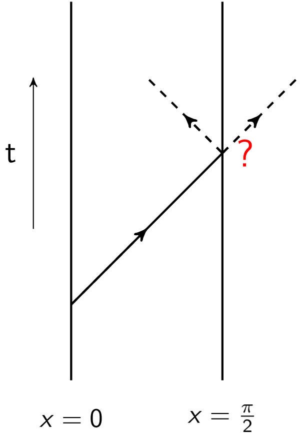
Anti-de Sitter spacetime in $d+1$ dimensions (AdS_{d+1})

- AdS_{d+1} spacetime is the maximally symmetric solution of the vacuum Einstein equations $G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0$ with negative Λ

$$g = \frac{\ell^2}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x \, d\omega^2_{S^{d-1}}\right), \qquad \Lambda = -\frac{2}{d(d-1)\ell^2}$$

where $0 \leq x < \pi/2$ and $-\infty < t < \infty$.

- Spatial infinity $x = \pi/2$ is the timelike cylinder $\mathcal{I} = \mathbb{R} \times S^{d-1}$ with the boundary metric $ds^2_{\mathcal{I}} = -dt^2 + d\omega^2_{S^{d-1}}$
- AdS is **not globally hyperbolic** - to make sense of evolution one needs to choose boundary conditions at $\mathcal{I}$
- Asymptotically AdS spacetimes by definition have the same conformal boundary as AdS



Is AdS stable?

- By the positive energy theorem AdS space is the unique ground state among asymptotically AdS spacetimes (much as Minkowski space is the unique ground state among asymptotically flat spacetimes)
- Basic question for any equilibrium solution: **do small perturbations of it at $t = 0$ remain small for all future times?**
- Key difference between Minkowski and AdS: the main mechanism of stability of Minkowski - **dissipation of energy by dispersion** - is absent in AdS (for no flux boundary conditions $\mathcal{I}$ acts as a mirror)
- The problem seems tractable only in spherical symmetry; we need matter to generate dynamics
- Simple matter model: massless scalar field

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G \left(\partial_\alpha \phi \, \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} (\partial\phi)^2 \right)$$

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = 0$$

- Convenient parametrization of asymptotically AdS spacetimes

$$ds^2 = \frac{\ell^2}{\cos^2 x}\left(-Ae^{-2\delta}dt^2 + A^{-1}dx^2 + \sin^2 x\, d\omega^2_{d-2}\right)$$

where A and δ are functions of (t, x).

- Define mass function $m(t,x) = \dfrac{\sin^{d-2} x}{\cos^d x}(1-A)$
- Field equations (using $8\pi G = d-1$ and $' = \partial_x$, $\dot{} = \partial_t$)

$$\left(A^{-1}e^{\delta}\dot{\phi}\right)^{\cdot} = (\tan x)^{1-d}\left(\tan^{d-1} x\, A\, e^{-\delta}\phi'\right)'$$

$$m' = (\tan x)^{d-1} A\, S, \quad \delta' = -\sin x \cos x\, S, \quad S := A^{-2}e^{2\delta}\dot{\phi}^2 + \phi'^2$$

- Initial-boundary problem is locally well-posed under the following boundary conditions near $x = \pi/2$ (Holzegel-Smulevici 2011)

$$\phi \sim \left(\frac{\pi}{2} - x\right)^d, \quad \delta \sim \left(\frac{\pi}{2} - x\right)^{2d}, \quad 1 - A = \left(\frac{\pi}{2} - x\right)^d$$

- We are interested in the long time evolution of small smooth perturbations of AdS spacetime $\phi = 0, m = 0, \delta = 0$.

Conjecture (B-Rostworowski 2011)

AdS_4 is unstable against the formation of a black hole for a large class of arbitrarily small perturbations

Evidence:

- **Perturbative:** due to the nondispersive character of the linear spectrum, resonant interactions between harmonics give rise to secular terms at higher orders of the formal perturbation expansion. This **shifts the energy spectrum to higher frequencies**. The same happens for vacuum Einstein equations (Dias-Horowitz-Santos 2011).
- **Heuristic:** the transfer of energy to higher frequencies (eo ipso, concentration of energy on finer and finer spatial scales) is expected to be eventually cut-off by horizon formation.
- **Numerical:** perturbations of size ε start growing after time $\mathcal{O}(\varepsilon^{-2})$. Subsequent nonlinear evolution leads to black hole formation (confirmed independently by Buchel-Lehner-Liebling 2012).

Generalization to $d \geq 4$ is straightforward (Jałmużna-Rostworowski-B).

AdS gravity in $d = 2$

- Spectral properties and nonlinear perturbation analysis are qualitatively the same in all dimensions $d \geq 3$
- Dimensionless measure of gravity's strength is GM/L^{d-2} so for $d = 2$ the *total* mass matters (not its concentration) ($d = 2$ **is the critical dimension for Einstein's equations**)
- AdS-Schwarzschild family in $d = 2$ (using $r = \ell \tan x$)

$$g = -N\,dt^2 + N^{-1} dr^2 + r^2\,d\varphi^2\,, \qquad N = 1 - M + r^2/\ell^2$$

  There is a mass gap between AdS_3 and the lightest black hole:
  - $M = 0$ — AdS_3
  - $0 < M < 1$ — conical (naked) singularities (Staruszkiewicz 1963)
  - $M > 1$ — BTZ black holes (Bañados-Teitelboim-Zanelli 1992)
- Small perturbations of AdS_3 cannot evolve into BTZ black holes
- Numerical studies of the threshold for black hole formation has been pioneered by Pretorius and Choptuik 2000

Possible scenarios of evolution

- Field equations (using $e^{\beta} := Ae^{-\delta}$)

$$\left(e^{-\beta}\dot{\phi}\right)^{\cdot} = \frac{1}{\tan x}(\tan x\, e^{\beta}\phi')'$$
$$m' = \tan x\, A\,(e^{-2\beta}\dot{\phi}^2 + \phi'^2), \quad \beta' = 2\sin x\cos x\,\frac{m}{A}$$

- **Black hole formation is excluded for** $M = \lim_{x\to\pi/2} m(t,x) < 1$.

  Proof: $g^{\alpha\beta}\partial_\alpha r\,\partial_\beta r = A = 1 - m\cos^2 x > 0$.

- There remains a dichotomy:
  - (a) Global-in-time regularity
  - (b) Naked singularity formation

- We will give evidence against (b) using the **analyticity strip method** (Sulem-Sulem-Frisch 1983).

Analyticity strip method

- Let $u(t, x)$ be a solution of an evolution equation starting from real-analytic initial data and let $u(t, z)$ be its analytic extension to the complex z-plane.
- Typically $u(t, z)$ will have complex singularities. Let $z = x + i\rho$ be the location of the singularity closest to the real axis (hence ρ measures the width of the analyticity strip around the real axis).
- If $\rho(t)$ vanishes at some $t = T < \infty$, then the solution "blows up"; otherwise it is globally regular in time.
- Fourier coefficients of $u(t, x)$ behave for large k as

$$\hat{u}_k(t) \sim k^{-\alpha} \exp(-\rho k)$$

- Method: compute $\rho(t)$ by fitting an exponential decay to the tail of the numerically computed Fourier spectrum

Example

$$u_t = x u_x + \alpha u^2 , \quad u(0,x) = \frac{\varepsilon}{1+x^2}$$

$$u(t,x) = \frac{\varepsilon}{1 + e^{2t}x^2 - \alpha\varepsilon t}$$

$$\hat{u}(t,k) = \frac{\varepsilon\pi e^{-t}}{\sqrt{1-\varepsilon\alpha t}} H(k) \exp(-k \underbrace{e^{-t}\sqrt{1-\varepsilon\alpha t}}_{\rho(t)}) + (k \leftrightarrow -k)$$

- $\alpha > 0$: blowup at $x = 0$ in time $T = 1/\varepsilon\alpha$
- $\alpha = 0$: global regularity but

$$||u||^2_{\dot{H}^s} := \int_{-\infty}^{\infty} (\partial_x^s u)^2 dx = c_s \, e^{(2s-1)t}$$

L^2-asymptotic stability ($s = 0$) and instability for $s > 1/2$.

Spectral properties

- Linearized equation (Breitenlohner-Freedman 1982, Ishibashi-Wald 2004)

$$\ddot{\phi} + L\phi = 0, \quad L = -\frac{1}{\tan x}\,\partial_x\left(\tan x\,\partial_x\right)$$

  L is essentially self-adjoint on $L^2([0,\pi/2], \tan x\,dx)$.

- Eigenvalues and eigenvectors of L are $(k = 0, 1, \dots)$

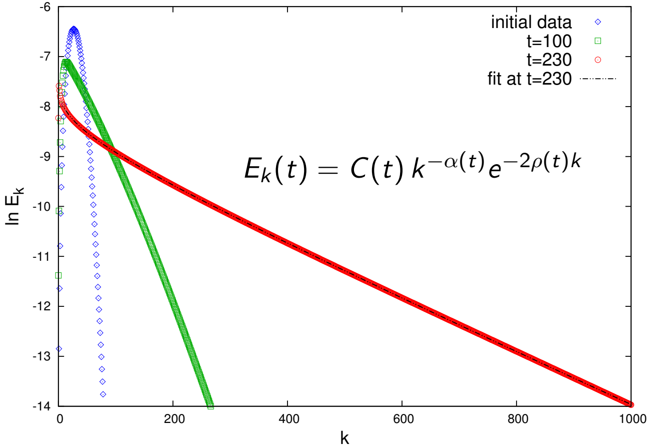
$$\omega_k^2 = (2+2k)^2, \quad e_k(x) = 2\sqrt{k+1}\cos^2 x\, P_k^{0,1}(\cos 2x)$$

- Inner product: $(f, g) = \int_0^{\pi/2} f(x)g(x)\tan x\,dx$
- Let us define $\phi_k := (\sqrt{A}\,\phi', e_k')$ and $p_k := (\sqrt{A}\,e^{-\beta}\dot{\phi}, e_k)$. Then

$$M = \int_0^{\pi/2} A\left(e^{-2\beta}\dot{\phi}^2 + \phi'^2\right)\tan x\,dx = \sum_{k=0}^{\infty} E_k(t)$$

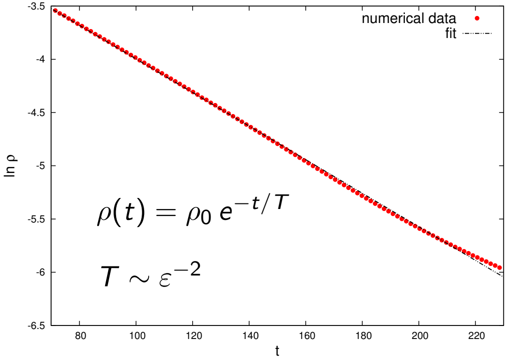
  where $E_k = p_k^2 + \omega_k^{-2}\phi_k^2$ is the energy of the k-th mode.

Computation of $\rho(t)$ from the energy spectrum



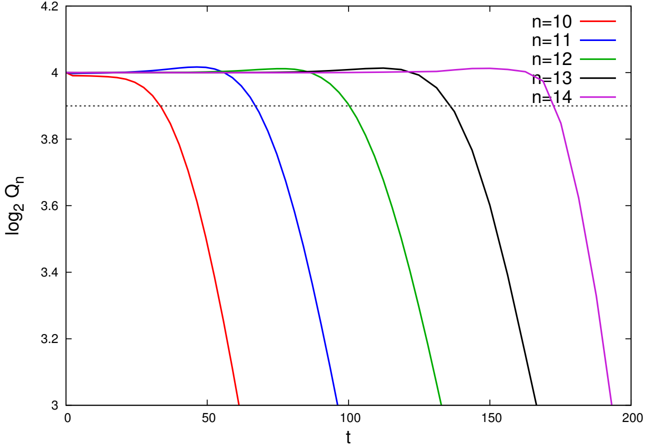
Initial data $\phi(0,x) = \varepsilon \exp(-\tan^2 x/\sigma^2)\,, \quad \dot{\phi}(0,x) = 0$

Evidence for global regularity



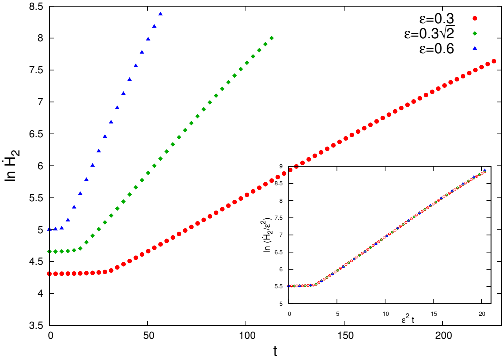
- Conjecture: as $t \to \infty$, solutions develop progressively finer spatial scales without ever losing smoothness (weak turbulence).
- Similar weakly turbulent loss of regularity occurs for the incompressible Euler equation in two dimensions (Yudovich 1974).

Convergence test



Convergence factor for the solution ϕ_n computed on the 2^n-grid is defined by $Q_n = \frac{||\phi_n - \phi_{n+1}||}{||\phi_{n+1} - \phi_{n+2}||}$, where $|| \cdot ||$ is the spatial ℓ_2-norm.

$\dot{H}^s$-instability for $s > 1$



Time evolution of the upper envelope of $\dot{H}^2 = ||\phi''(t,x)||_2$

- AdS$_3$ is $\dot{H}^s$-unstable for arbitrarily small perturbations (for $s > 1$).
- The turbulent instability is not active for some perturbations - time-periodic solutions (see Andrzej's talk).

Final remarks

- Weak turbulence is expected to be common for nonlinear wave equations in bounded domains.
- In the case of Einstein's equations, **the weakly turbulent dynamics can proceed forever only in $d = 2$**, whereas in higher dimensions it is unavoidably cut off in finite time by the black hole formation.
- The nature of the threshold at $M = 1$ is not well understood. Does every solution with $M > 1$ evolve into a black hole?
- **Finite energy threshold for blowup is typical for nonlinear wave equations in critical dimensions** (for instance, wave maps in $d = 2$ or Yang-Mills equations in $d = 4$).
- Are there any interesting holographic implications of weak turbulence?